Rigorous bound on the plane-shear-flow dissipation rate

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The Doering-Constantin approach [Phys. Rev. Lett. **69**, 1648 (1992)] to derive an upper bound on the dissipation rate in terms of a properly chosen stationary flow profile can be (slightly) improved by casting it in the form of a variational principle for the profile. Only Schwarz's inequality is needed to prove this. Its solution can be evaluated analytically. The mean dissipation bound is $\varepsilon \le U^3L^{-1}/9\sqrt{2}$ provided the Reynolds number Re$=UL/\nu$ exceeds $12\sqrt{2}$, and the mean transverse momentum flux is bounded by $J_{31} \le (\nu L^{-1})^2 \mathrm{Re}^2/9\sqrt{2}$.

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I. INTRODUCTION

Turbulent flow is a very complex dynamical system. Therefore, rigorous results based on the Navier-Stokes equations are not obtained too often. It was quite recently that Doering and Constantin [1] succeeded in deriving a rigorous upper bound for the Reynolds number dependence of the dissipation rate in plane-shear flow. The idea is to substitute the velocity field by a chosen flow, whose dissipation is shown to bound that of the real flow. The chosen flow is different from the average profile but also time independent and one-dimensional only.

The question arises if the bound can be improved by optimizing the chosen flow. Improvement seems possible since the bound is still considerably larger than measured dissipation rates (although measured [2] in another flow, namely, in Taylor-Couette flow). A variational approach for the dissipation bound was proposed in [3], which seems to capture dynamical aspects of the representative flow. It is much more elaborate.

In the present paper, we stay within the simpler frame of admitting stationary one-dimensional profiles as trial functions, but optimize their choice. This results in a minimum principle with a profile constraint. It, thus, is a stationary counterpart of the more dynamical principle proposed in Ref. [3]. We derive our minimum principle by using Schwarz's inequality only. The proof is rather simple and hopefully provides some insight into the quality of the bound. The ansatz closely follows [1]. Because of its simplicity, our approach might be extendable to more complicated geometries. Due to the variational form, we exhaust the estimate within the given stationary approach.

II. MOMENTUM FLUX AND DISSIPATION

Consider the three-dimensional flow $\mathbf{u}(\mathbf{x},t)$ between two infinitely extended parallel planes (Fig. 1), one at rest and the other one moving with velocity U in x-direction. The laminar flow profile corresponding to the shear U/L is $\mathbf{U}_{\mathrm{lam}} = \mathbf{e}_x Uz/L$. Here, L denotes the distance between the planes and $0 \le z \le L$. Because of the shear there is momentum flux. This can be calculated by averaging the Navier-Stokes equations over x-y planes.

$$\partial_t u_i(\mathbf{x},t) = -\mathbf{u}\cdot\boldsymbol{\nabla} u_i - \partial_i p + \nu\Delta u_i, \quad i=1,2,3 \quad , \tag{1}$$

$$\operatorname{div}\mathbf{u} = 0, \quad \text{incompressibility} \; , \tag{2}$$

$$\mathbf{u}(z=0)=\mathbf{0}, \quad \mathbf{u}(z=L) = U\mathbf{e}_x, \qquad \text{boundary conditions} \; . \tag{3}$$

Here, $p(\mathbf{x},t)$ is the kinematic pressure (i.e., the physical pressure divided by the mass density ρ) and ν is the kinematic viscosity. x-y-plane averaging $\langle\cdots\rangle_A$ and averaging on time (which is understood as included in $\langle\cdots\rangle_A$) imply for $i=1$ the equation $0 = -\partial_3\langle u_3u_1\rangle_A + \nu\partial_3\partial_3\langle u_1\rangle_A$. Therefore,

$$J_{31} \equiv \langle u_3 u_1\rangle_A - \nu\partial_3\langle u_1\rangle_A = \text{const (in } z) \tag{4}$$

is the conserved transverse momentum current density. In the laminar case $J_{31} = -\nu U/L$ and $J_{32}=0$. For turbulent flow the viscous term dominates in the vicinity of the boundaries, whereas in the bulk mainly the correlation $\langle u_3u_1\rangle_A$ carries the momentum flux.

There is an exact relation between the transverse momentum flux and the dissipation rate, the latter one av-

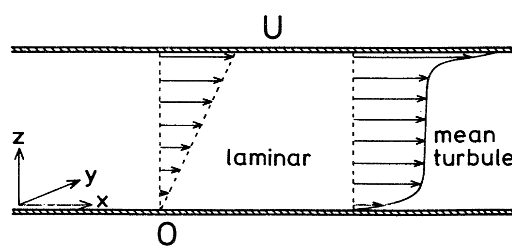

FIG. 1. Plane-shear flow, laminar and mean turbulent profiles, schematic.

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eraged over the total flow volume. Multiply (1) by u_i, volume average $\overline{(\cdots)}$ (as well as time average),

$$\overline{\nu u_{i|j} u_{i|j}} \equiv \varepsilon = -\oint \frac{d\mathbf{A}}{V}\left[\left(\frac{u^2}{2}+p\right)\mathbf{u} - \nu \boldsymbol{\nabla}\frac{u^2}{2}\right] . \quad (5)$$

For plane-shear flow on the right-hand side only, the ν term contributes and only at the upper (sheared) plane,

$$\varepsilon = \nu U L^{-1} \partial_3 \langle u_1 \rangle_{A,z=L} = -U L^{-1} J_{31} . \quad (6)$$

ε also equals the mechanical power necessary to maintain the shear [calculate either $(F_{\|}/\rho A L)U$ or be aware that it is the rate of energy loss]. Note that ε is the global volume average of the dissipation rate, i.e., including the boundary layers. Determining the time average of $\nu u_{i|j}^2(\mathbf{x},t)$ from velocity signals $\mathbf{u}(\mathbf{x},t)$ measured in the turbulent bulk will give an average bulk dissipation rate ε_B that may differ considerably from ε defined in equation (5).

In contrast, the transverse momentum flux can be measured also in the bulk as the transverse velocity correlation $\overline{u_3 u_1}$. One can nondimensionalize the transverse momentum flux by expressing it in terms of the laminar one, $\nu U L^{-1}$, or in terms of the dissipative velocity νL^{-1} squared. The former choice gives the momentum flux analogon of the Nusselt number Nu describing heat flux,

$$\mathrm{Nu}_{31} = J_{31}/\nu U L^{-1} = \mathrm{Re}\, \varepsilon / U^3 L^{-1} . \quad (7)$$

Relation (6) was used. The latter choice leads to

$$\widetilde{\mathrm{Nu}}_{31} = J_{31}/(\nu L^{-1})^2 = \mathrm{Re}^2 \varepsilon / U^3 L^{-1} . \quad (8)$$

Both "momentum flux Nusselt numbers" are thus given in terms of the dimensionless globally averaged dissipation rate

$$\varepsilon / U^3 L^{-1} \equiv c_\varepsilon(\mathrm{Re}), \quad (9)$$

where Re is the Reynolds number.

For the laminar plane-shear profile one gets

$$c_\varepsilon(\mathrm{Re}) = 1/\mathrm{Re} \ , \ \text{laminar} . \quad (10)$$

If the flow becomes turbulent, there is increased dissipation and $c_\varepsilon(\mathrm{Re})$ will be larger than 1/Re. It is shown already in [4] (cf. also [5]) and again in [1] that 1/Re is always a rigorous lower bound for $c_\varepsilon(\mathrm{Re})$. Our aim here is to obtain an upper bound for the dimensionless globally averaged dissipation rate $c_\varepsilon(\mathrm{Re})$. This in turn will lead to the momentum flux bound displayed in Fig. 2.

The Reynolds number dependence of the dimensionless bulk dissipation rate $\varepsilon_B / u'^3 L^{-1} \equiv c_{\varepsilon,B}$ and its crossover from the laminar (1/Re) behavior to the constant level for fully developed turbulence has recently been derived by Lohse [6]. The effects of intermittency on $c_{\varepsilon,B}(\mathrm{Re})$ have been analyzed in [7]. Experiments are reported in [8] and, in particular, the effect of shear on the bulk dissipation in [9]. Exact relations between global dissipation and flux are also known for thermally driven turbulence [10]; in that case, also, the total volume averages differ appreciably from the locally averaged bulk values [11]. In the Rayleigh-Bénard flow

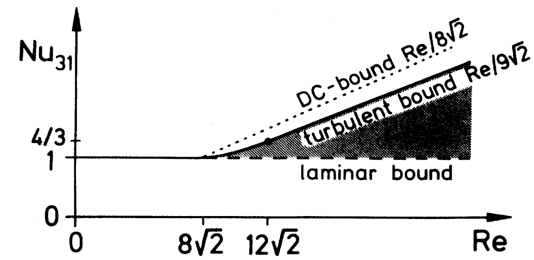

FIG. 2. Dimensionless transverse momentum flux bounds. DC bound indicates the Doering-Constantin bound [1], the new bound results from (7) with $\varepsilon/U^3L^{-1} = c_\varepsilon(\mathrm{Re})$ from (31).

$$\varepsilon = \nu \overline{u_{i|j}^2} = \nu \kappa^2 L^{-4}(\mathrm{Nu}-1)\mathrm{Ra} ,$$
$$\varepsilon_\theta = \kappa \overline{\theta_{|i}^2} = \kappa L^{-2} \Delta^2 \mathrm{Nu} .$$

κ is the temperature diffusivity, Δ the external temperature difference, Ra the Rayleigh number, Nu the Nusselt number, and ε_θ the temperature dissipation.

III. VARIATIONAL PRINCIPLE

Now the variational principle for a time-independent flow profile is derived, whose dissipation rate bounds the global ε of the real flow. Now, decompose the physical flow field into two parts,

$$\mathbf{u}(\mathbf{x},t) = \mathbf{U}(\mathbf{x}) + \mathbf{v}(\mathbf{x},t) . \quad (11)$$

The **U** field will be chosen properly and $\mathbf{v}(\mathbf{x},t)$ is defined by (11). This decomposition leads to the dissipation rate bound,

$$\varepsilon = \nu\left(\overline{U_{i|j}^2} + 2\overline{U_{i|j} v_{i|j}} + \overline{v_{i|j}^2}\right) \le \nu \overline{U_{i|j}^2}, \quad (12)$$

provided **U** is chosen such that the sum of the last two terms is negative. The properties which **U** shall have are now listed: (i) time independent, $\mathbf{U}(\mathbf{x})$; (ii) incompressible, div $\mathbf{U} = 0$; (iii) satisfying the same boundary conditions as **u**; (iv) parallel to mean flow, $\mathbf{U} = U_1 \mathbf{e}_x$; (v) depending on height z only, $U_1(z)$; (vi) symmetry with respect to $z = L/2$; (vii) satisfying $\overline{v_{i|j}^2} + 2\overline{v_{i|j} U_{i|j}} \le 0$ for any **v**, "profile constraint."

With the ansatz

$$\mathbf{U} = \mathbf{e}_x U_1(z) = \mathbf{e}_x U f(z/L) , \quad (13)$$

$$f(0) = 0 \ , \ f(1) = 1 , \quad (14)$$

the conditions (i) through (v) are satisfied. Because of the symmetry (vi), we can restrict ourselves to $0 \le \xi \le 1/2$ and can write

$$f(0) = 0 \ , \ f(1/2) = 1/2, \quad (15)$$

instead of (14). By the choice (vi) the symmetry of the physical flow field **u** is extended to the **v** field.

It is now easy to calculate the ε bound by inserting (13) [i.e., $U_{1|3} = UL^{-1}f'(\xi)$ with $\xi = z/L$] into the rhs of (12),

$$\varepsilon/U^3L^{-1} = c_\varepsilon \leq \min_f 2\mathrm{Re}^{-1} \int_0^{1/2} d\xi\, f'(\xi)^2 \,. \qquad (16)$$

The profile functions $f(\xi)$ have to satisfy the boundary conditions (15) and have to respect the profile constraint resulting from (vii). We shall show that the inequality,

$$\int_0^{1/2} d\xi\, \xi |f'(\xi)| \leq \sqrt{2}/\mathrm{Re} \,, \qquad (17)$$

for the trial profile $f(\xi)$ is a sufficient condition to satisfy the profile constraint (vii) under the specifications (i) through (vi). In addition, some not necessarily sharp estimates for the $\mathbf{v}$ field will be made. That means, this inequality (17) does not exhaust the original profile constraint. We, therefore, denote it as the "restricted profile constraint" for $\mathbf{U}$.

Before proving (17) as the restricted profile constraint, we remark that the special choice of a trial field $\mathbf{U}$ used in [1], namely, a linear profile near the boundaries and constant, z-independent flow in the center range, satisfies (13), (15); the only free parameter is the thickness δ of the linear boundary range:

$$f(\xi) = \xi/2\delta \,,\ 0 \leq \xi \leq \delta\,(\leq 1/2) \quad ,$$
$$f(\xi) = 1/2 \,,\ \delta \leq \xi \leq 1/2 \,. \qquad (18)$$

The profile constraint (17) leads to $\delta \leq 4\sqrt{2}/\mathrm{Re}$. Since δ is 1/2 at most, this only implies a restriction on δ if $\mathrm{Re} \geq 8\sqrt{2}$. For c_ε, one immediately gets

$$c_\varepsilon(\mathrm{Re}) \leq \begin{cases} 1/\mathrm{Re}\,, & 0 \leq \mathrm{Re} \leq 8\sqrt{2}\,, \\ 1/8\sqrt{2}\,, & 8\sqrt{2} \leq \mathrm{Re} < \infty\,. \end{cases} \qquad (19)$$

This bound, first given by Doering and Constantin [1], can be improved by avoiding the special choice (18). Considering (16) as a minimum principle with the constraint (17) and the boundary conditions (15) allows us to optimize the profile $U_1 = Uf(z/L)$.

IV. PROOF

The proof of the restricted profile constraint (17) can be given by repeatedly using Schwarz's inequality. The first step is an equality still.

$$\overline{v_{i|j}^2} + 2\overline{v_{i|j}U_{i|j}} = -\overline{v_{i|j}^2} - \frac{2}{\nu}\overline{v_iv_jU_{i|j}} \qquad (20)$$

from multiplying the equation of motion for $\partial_t v_i = \partial_t u_i = \cdots$ by v_i and volume integrating. By incompressibility, vanishing of $\mathbf{v}$ at the boundaries and stationarity of $\overline{\mathbf{v}^2/2}$ one obtains after integrating by parts $\overline{v_iv_jU_{i|j}} + \nu\overline{v_{i|j}^2} + \nu\overline{v_{i|j}U_{i|j}} = 0$, i.e., Eq. (20). [The properties (iv) and (v) imply that $U_jU_{i|j}$ does not contribute.]

Next, we estimate the second term in (20). Let A denote the area of the parallel planes ($A \to \infty$), the volume is $V = AL$.

$$\begin{aligned}\left|\frac{2}{\nu}\overline{v_iv_jU_{i|j}}\right| &= \frac{2}{\nu}\frac{1}{AL}\left|2\int_0^{L/2}dz\int_A dA(x,y)\,v_1v_3U\frac{df(z/L)}{dz}\right| \\ &= \frac{4U}{\nu AL}\left|\int_0^{1/2}d\xi\int_A dA\,v_1v_3f'(\xi)\right| \\ &\leq \frac{4U}{\nu AL}\int_0^{1/2}d\xi\,|f'(\xi)|\int_A\left|\int_0^{\xi L}v_{1|3}(z')dz'\right|\left|\int_0^{\xi L}v_{3|3}(z')dz'\right|dA\,.\end{aligned}$$

Note

$$\left|\int_0^{\xi L} v_{1|3}(z')dz'\right| \leq \left(\int_0^{\xi L} dz'\,1^2 \int_0^{\xi L} v_{1|3}^2(z')\,dz'\right)^{1/2} \,,$$

etc., leading to

$$\left|\frac{2}{\nu}\overline{v_iv_jU_{i|j}}\right| \leq 2\mathrm{Re}\left(\overline{v_{1|3}^2}\,\overline{v_{3|3}^2}\right)^{1/2}\int_0^{1/2}d\xi\,\xi|f'(\xi)| \,. \qquad (21)$$

There was no need to specialize the profile $f(\xi)$, $\xi = z/L$. Next, the v derivatives can be bounded by

$$\left(\overline{v_{1|3}^2}\,\overline{v_{3|3}^2}\right)^{1/2} \leq \overline{v_{i|j}^2}/(2\sqrt{2}) \,. \qquad (22)$$

Start with incompressibility $v_{1|1} + v_{2|2} + v_{3|3} = 0$ and get $\overline{v_{3|3}^2} = \overline{v_{1|1}^2} + 2\overline{v_{1|1}v_{2|2}} + \overline{v_{2|2}^2}$ and $2\overline{v_{3|3}^2} = \overline{v_{3|3}^2} + \overline{v_{3|3}^2} \leq \overline{v_{3|3}^2} + \overline{v_{1|1}^2} + \left(\overline{v_{1|2}^2} + \overline{v_{2|1}^2}\right) + \overline{v_{2|2}^2}$. Also, from $ab \leq (a^2+b^2)/2$, find $\sqrt{\overline{v_{1|3}^2}}\sqrt{\overline{v_{3|3}^2}} = \sqrt{\overline{v_{1|3}^2}}\sqrt{2\overline{v_{3|3}^2}}/\sqrt{2} \leq \left(\overline{v_{1|3}^2} + 2\overline{v_{3|3}^2}\right)/2\sqrt{2}$, leading directly to the derivative bound (22). It is a safe though most probably not exhausted estimate. Finally, the restricted profile constraint (17) follows by keeping the magnitude of the second term in (20) below that of the first one, utilizing the inequality (21).

V. SOLUTION

The minimum principle (16) for the dissipation rate together with the boundary conditions (15) and the profile constraint (17) can be solved analytically. Start by converting the inequality of the restricted profile constraint into an equality,

$$\int_0^{1/2} d\xi\, \xi\, |f'(\xi)| = \frac{a\sqrt{2}}{\text{Re}} \quad , \quad 0 \le a \le 1 \, . \tag{23}$$

In this form the constraint can be included in (16) with a Lagrangean multiplier λ. In the resulting $c_\varepsilon(\text{Re}, a)$ one optimizes with respect to a. Next, since $f(\xi)$ has to grow from 0 at the boundary to 1/2 in the center, we expect f' to be positive. Intervals of negative f' would imply others with even larger $f' > 0$, so leading to larger c_ε from (16). Therefore, we look for $f'(\xi) \ge 0$, all ξ, and skip the modulus signs $|\,|$. The Euler-Lagrange equation of the variational principle then reads

$$f''(\xi) - \text{Re}\lambda/4 = 0 \, . \tag{24}$$

The boundary conditions $f(0) = 0$ and $f(1/2) = 1/2$ determine the two integration constants. The resulting profile $f(z/L)$ is

$$f(\xi) = \frac{\text{Re}\lambda}{8}\xi^2 + \left(1 - \frac{\text{Re}\lambda}{16}\right)\xi \, . \tag{25}$$

Inserting (25) into the constraint (23) fixes the Lagrangean multiplier,

$$\lambda(\text{Re}, a) = \frac{48}{\text{Re}}\left(\frac{8\sqrt{2}a}{\text{Re}} - 1\right) \, . \tag{26}$$

This solves the minimum principle, provided $f'(\xi)$ stays positive as was assumed.

$$f'(\xi) = 12\xi\left(\frac{8\sqrt{2}a}{\text{Re}} - 1\right) + 4\left(1 - \frac{6\sqrt{2}a}{\text{Re}}\right) \ge 0 \, . \tag{27}$$

As long as Re is small enough, we can minimize $c_\varepsilon(\text{Re}, a)$ with $a = \text{Re}/8\sqrt{2}$, thus $f' = 1$ and $\lambda = 0$. This is possible for $0 \le \text{Re} \le 8\sqrt{2}$. When Re exceeds $8\sqrt{2}$ one has to choose $a = 1$. To now guarantee $f' \ge 0$ even at $\xi = 1/2$, the Reynolds number must be [from solving (27)] $\text{Re} \le 12\sqrt{2}$. In the range $8\sqrt{2} \le \text{Re} \le 12\sqrt{2}$ the profile is quadratic in ξ (since now $\lambda \ne 0$, negative) and $f'(1/2)$ decreases from 1 down to 0, while $f'(0)$ steepens from 1 to 2, i.e., a boundary layer develops.

If Re is even larger, $\text{Re} \ge 12\sqrt{2}$, $f'(1/2)$ would become negative. We, therefore, have to consider a smaller interval $0 \le \xi \le \delta$, with $\delta \le 1/2$, and $f(\delta) = 1/2$. The profile function f stays constant from δ to 1/2. This boundary condition leads [by solving again the Euler-Lagrange equation (24)] to

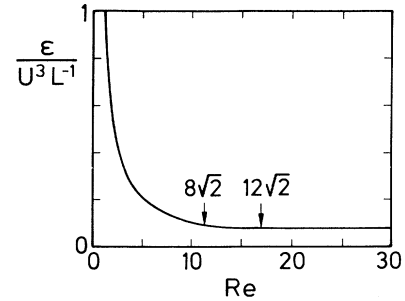

FIG. 3. Rigorous bound on the total volume average of the dissipation rate ε/U^3L^{-1} versus Reynolds number $\text{Re} = UL/\nu$. The arrows indicate the borders $8\sqrt{2}$ and $12\sqrt{2}$ between the ranges laminar (I), developing boundary layer (but still filling the whole cross section) (II), and turbulent (III).

$$f(\xi) = \frac{\text{Re}\lambda}{8}\xi^2 + \left(\frac{1}{2\delta} - \frac{\text{Re}\lambda\delta}{8}\right)\xi \quad , \quad 0 \le \xi \le \delta \, , \; \delta \le \frac{1}{2} \, . \tag{28}$$

As before, the multiplier λ is obtained from the profile constraint, $\text{Re}\lambda\delta^2 = 12(4\sqrt{2}/\text{Re}\delta - 1)$. The condition $f'(\delta) = 0$ implies $\text{Re}\lambda\delta^2 = -4$. Combining both gives

$$\delta = 6\sqrt{2}/\text{Re} \le 1/2 \, , \tag{29}$$

since $\text{Re} \ge 12\sqrt{2}$. δ is the thickness of the optimal boundary layer. The profile of the boundary layer is of second order, $f(\xi) = -(\text{Re}^2/144)\xi^2 + (\text{Re}/6\sqrt{2})\xi$, $0 \le \xi \le \delta \le 1/2$, as λ turns out to be $\lambda = -\text{Re}/18$.

We now have calculated the optimal profiles as controlled by Re and can easily determine $c_\varepsilon(\text{Re})$ from (16) with the respective $f'(\xi)$ in the different Re ranges. There are three such ranges.

(I) $0 \le \text{Re} \le 8\sqrt{2}$, laminar range,

(II) $8\sqrt{2} \le \text{Re} \le 12\sqrt{2}$,

development of boundary layer, (30)

(III) $12\sqrt{2} \le \text{Re} < \infty$, turbulence range.

Within these ranges $c_\varepsilon(\text{Re})$ and the corresponding profiles $f(\xi;\text{Re})$ can be summarized as follows:

$$\frac{1}{\text{Re}} \le c_\varepsilon(\text{Re}) \le \begin{cases} 1/\text{Re}, & \text{I} \\ 4(96\text{Re}^{-2} - 12\sqrt{2}\text{Re}^{-1} + 1)/\text{Re}, & \text{II} \\ 1/9\sqrt{2}, & \text{III} \, . \end{cases} \tag{31}$$

$$\frac{U_1(z/L)}{U} = f(\xi;\text{Re}) = \begin{cases} \xi, & \text{I} \\ 6(8\sqrt{2}\text{Re}^{-1} - 1)\xi^2 + 4(1 - 6\sqrt{2}\text{Re}^{-1})\xi, & \text{II} \\ -(\text{Re}^2/144)\xi^2 + (\text{Re}/6\sqrt{2})\xi, & \text{III} \; ; \end{cases} \tag{32}$$

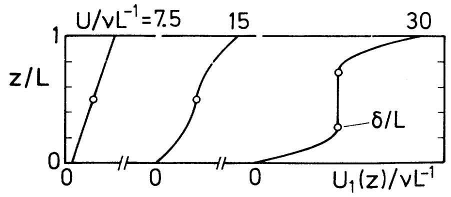

FIG. 4. Optimal time-independent profiles in the ranges laminar (I), boundary layer developing (II), turbulent (III). The corresponding Re are 7.5 ($= 5.3\sqrt{2}$), 15 ($= 10.6\sqrt{2}$), and 30 ($= 21.2\sqrt{2}$). All profiles are $f(\xi) \in C^1[0,1]$. The thickness of the boundary layer decreases as $\delta = 6\sqrt{2}/\text{Re}$.

the profile given in III holds for $0 \leq \xi \leq \delta$, whereas $f(\xi) = 1/2$ is constant for $\delta \leq \xi \leq 1/2$.

This dissipation bound (31) and some optimal profiles are displayed in Figs. 3 and 4. The optimal profile develops a boundary layer whose thickness δ decreases $\propto \text{Re}^{-1}$. But note that $U_1(z/L) = Uf(\xi)$ is *not* the time mean of the real physical turbulent flow but instead the optimum for the ε/U^3L^{-1}-bound under the provisos (i) through (vi) and (17). In [4,5], an optimal profile was obtained which still has a finite shear also in the bulk. The provisos were more general, allowing for y-dependent structure in the optimal field instead of our choice (v). The approach suggested in [4,5] is asymptotic in the Reynolds number, while the bound (31), here, is rigorous for all Re (as in [1]).

VI. DISCUSSION

The effect of optimizing the profile instead of simply choosing a linear ansatz (as in [1]) is only moderate. The second order profile (32) improves the bound from $1/8\sqrt{2}$ to $1/9\sqrt{2}$ obtained here. Within the conditions (i) through (vi) and the restricted profile constraint (17) this seems to exhaust the method of a stationary trial profile which depends only on height z.

If one allows for a y dependence in addition, a somewhat different approach (the optimum theory introduced by Busse [4,5], which gives asymptotic ($\text{Re} \to \infty$) bounds) leads to an upper bound for c_ε which according to Fig. 1 in Ref. [4(b)] is by about a factor of 4 smaller than (31), but also Re independent. I.e., the corresponding transverse momentum flux also is $\widetilde{\text{Nu}}_{31} \propto \text{Re}^2$.

The Re-independent bound $1/9\sqrt{2}$ ($= 0.079$) for ε/U^3L^{-1} has to be compared with the results in [6,7] that ε_B/u'^3L^{-1} approaches the constant 0.6 (or, according to data [8] a value near 1). Both well compare if, e.g., $\varepsilon_B \approx \varepsilon$ and $u' \approx U/2$ or if $u' \approx U/3$ and $\varepsilon_B \approx 0.3\,\varepsilon$.

According to Ref. [7] the presence of scaling corrections in the exponent of the structure function $D(r)$ due to intermittency leads to a power law decrease of $c_{\varepsilon,B}(\text{Re})$ with increasing Re. One expects a similar behavior of $c_\varepsilon(\text{Re})$. Therefore, if we introduce the scaling exponent β by $\text{Nu}_{31} \propto \text{Re}^\beta$, the value $\beta = 1$ corresponds to a Re independent c_ε as found for nonintermittency corrected $D(r)$ (cf. [6]), whereas $\beta < 1$ corresponds to intermittency corrections in the scaling behavior of turbulent flow (cf. [7]). Our bound (31) corresponds to classical, nonintermittent scaling as described by the Kolmogorov-Obukhov theory. Precise measurements of β would allow to identify scaling corrections. According to recent work, β might be less than 1 in an intermediate Re range but is expected to approach unity asymptotically with $\text{Re} \to \infty$, see [12–15]. The data reported in [2] (see also [1]) seem to be compatible with this scenario.

An asymptotically constant dimensionless dissipation rate c_ε means that the transverse momentum flux Nusselt number (7) scales as $\text{Nu}_{31} \propto \text{Re}$. To be more precise, we have [from (7)] $\text{Nu}_{31} \leq \text{Re}/9\sqrt{2}$ or the transverse momentum flux bound,

$$J_{31} \leq U^2/9\sqrt{2}\,. \tag{33}$$

The linear scaling $\text{Nu}_{31} \propto \text{Re}$ [or the explicit expression (33)] is equivalent to an L independence of the transverse momentum flux J_{31}. Namely, $\text{Nu}_{31} \propto \text{Re}^\beta$ together with $\text{Nu}_{31} = J_{31}/\nu UL^{-1}$ and L independent J_{31} implies $\beta = 1$ by comparison of L exponents. Thus, any deviation of the Re scaling exponent of Nu_{31} from unity, which is not excluded yet by the experiments (cf. [2,16]) implies that the transverse momentum flux will depend on the channel width L, however large L may be.

This scaling argument can be generalized to include the boundary layers of thickness δ and the velocity fluctuations u' in the bulk, if similar scaling arguments are used as introduced in [17] to describe thermally driven (Rayleigh-Bénard) flow. Let $\text{Nu}_{31} \propto \text{Re}^\beta$, boundary layer thickness $\delta/L \propto \text{Re}^{-\alpha}$, and bulk velocity fluctuations $u'/\nu L^{-1} \propto \text{Re}^\gamma$. Then, relate the boundary layer with the bulk by continuity of transverse momentum flux. (a) In the viscous boundary layer it is $J_{31} \sim \nu U/\delta$, thus $\text{Nu}_{31} \sim L/\delta$, i.e., $\beta = \alpha$. (b) In the bulk $J_{31} \sim (u')^2$, i.e., $\text{Nu}_{31} \sim \text{Re}^{2\gamma-1}$ or $\beta = 2\gamma - 1$. (c) The dissipation term of the Navier-Stokes equation in the bulk has to be balanced by the nonlinear transport, $\nu u'/\eta^2 \sim (u')^2/\eta$. Provided δ is of the order of the Kolmogorov length η, this implies $u'\delta/\nu \sim 1$ and, thus, $\gamma = \alpha\,(= \beta)$. From (a), (b), and (c) one concludes $\beta = \alpha = \gamma = 1$. This means, consistent with the discussion before, that $\text{Nu}_{31} \propto \text{Re}$, $c_\varepsilon \propto \text{Re}^0$, $\delta/L \propto \text{Re}^{-1}$, $u' \propto \nu L^{-1}\text{Re} = U$. Again, we get "classical" scaling in agreement with the rigorous bound c_ε from Eq. (31) in range III.

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